2019-09-24 (pre-lecture) MATB44-2017 Prof. Yun William Yu

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Outline:

0. Norms and Fixed points
1. Proof. Contraction $\rightarrow$ fixed pt
2. Examples of contractions
3. Picard iteration

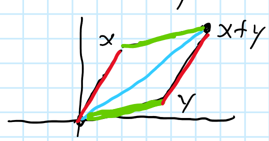
Last time:

We gave examples of norms, fixed points and contractions.

A **norm** on a vector space X has to satisfy 3 properties

1) $\|0\|=0$, $\|x\|>0$ for $x\neq 0$
2) $\|\alpha x\|=|\alpha|\,\|x\|$ for $\alpha\in\mathbb{R}$ and $x\in X$
3) $\|x+y\|\leq\|x\|+\|y\|$ for $x,y\in X$.

Ordinary triangle inequality



$\|x+y\|\leq\|x\|+\|y\|$

This time:

Claim: Also implies inverse triangle inequality $\big|\|x\|-\|y\|\big|\leq\|x-y\|$.

proof. $\|x\|=\|(x-y)+y\|\leq\|x-y\|+\|y\|$

$$\|x\|-\|y\|\leq\|x-y\|.$$

Similarly $\|y\|-\|x\|\leq\|y-x\|=\|x-y\|$

$$\Rightarrow \big|\|x\|-\|y\|\big|\leq\|x-y\|.$$

Examples of norms

Ex. • $X=\mathbb{R}$, $\|x\|=|x|$.

1) $|0|=0$, $|x|>0$ if $x\neq 0$.
2) $|\alpha x|=|\alpha|\,|x|$.
3) $|x+y|\leq|x|+|y|$.

thus, absolute value is a norm in $\mathbb{R}$

Ex. • $X=C(I)$, I a compact interval on $\mathbb{R}$ (e.g. $I=[0,1]$).

$$\|x\|=\sup_{t\in I}|x(t)|.$$

Note that the **Supremum** is the least upper bound of a set.
i.e. Given a set $A \subseteq \mathbb{R}$, define $B = \{ b \mid b \geq a \ \forall a \in A \}$.
Then $\sup(A) = \min(B)$.

e.g. $A = (0,1)$. $\quad \sup(A) = 1$.
$A = [0,1]$. $\quad \sup(A) = 1$.
$A = \{0,1,2,3\}$ $\quad \sup(A) = 3$.
$A = \mathbb{R}$. $\quad \sup(A)$ is undefined. (or sometimes we say $\sup(A) = +\infty$)

A few properties:

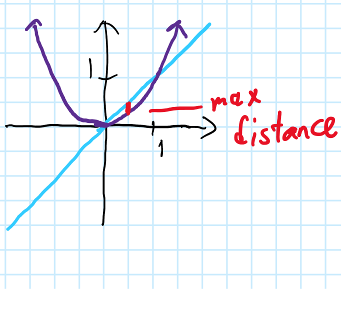
- $\sup(A+B) \equiv \sup(\{ a+b \mid a \in A,\ b \in B \}) = \sup(A) + \sup(B)$
- If $x: I \to \mathbb{R}$, $y: I \to \mathbb{R}$,
  $$\sup_{t \in I} (x(t) + y(t)) \leq \sup_{t \in I} x(t) + \sup_{t \in I} y(t)$$
- $\sup(\alpha A) \equiv \sup(\{ \alpha a \mid a \in A \}) = \alpha \sup(A)$.

Ex.
- $x(t) = t^2 + 1$, and $I = [0,1]$
  $$\|x\| = \sup_{t \in [0,1]} |t^2 + 1| = 2.$$
- $x(t) = -t - 5$, and $I = [0,1]$
  $$\|x\| = \sup_{t \in [0,1]} |-t-5| = 6.$$
- $x(t) = \sin(t) - 1$ and $I = [0, 1000000]$
  $$\|x\| = \sup_{t \in I} |\sin(t)| = 2$$

Note: This function $\|x\| = \sup_{t \in I} |x(t)|$ is a norm on $C(I)$. (HW)

This means we can use it to define "distances" between functions.
The "distance" between two functions is the maximum distance b/t them on the interval.

Ex. $x(t) = t^2$, $\quad y(t) = t$, $\quad I = [0,1]$



$$\|x - y\| = \sup_{t \in I} |t^2 - t|$$

Max distance is at a critical pt of $t^2 - t$.
Let $g(t) = t^2 - t$
$g'(t) = 2t - 1$
$\Rightarrow t = \frac{1}{2}$ is crit pt

$$g'(t) = 2t - 1$$
$$\Rightarrow t = \tfrac{1}{2} \text{ at crit. pt.}$$
$$g(\tfrac{1}{2}) = -\tfrac{1}{4}.$$

$$\Rightarrow \sup_{t \in I} |t^2 - t| = \frac{1}{4}.$$

A sequence of functions $x_n(t)$ **converges** to $x(t)$ if and only if
$$\lim_{n\to\infty} \|x_n - x\| = \lim_{n\to\infty} \sup_{t\in I} |x_n(t) - x(t)| = 0. \qquad \text{(uniform convergence)}$$

Turns out (real analysis) that a Cauchy sequence under this metric converges to another continuous function, so the space is **complete.**

Thus, the vector space $C(I)$, together with $\|x\| = \sup_{t\in I} |x(t)|$ (for I a compact interval) is a **Banach space.**

We can now reason about $C(I)$ using tools of Banach spaces.

Def. A **fixed point** of a mapping $K: C \subseteq X \to C$ is an element $x \in C$ s.t. $K(x) = x$.

Def. A **contraction** is a mapping $K: C \subseteq X \to C$ where there exists a contraction constant $\theta \in [0,1)$ s.t.
$$\|K(x) - K(y)\| \le \theta \|x - y\|, \qquad x, y \in C.$$

Notation: $K^n(x) = K(K^{n-1}(x))$, and $K^0(x) = x$.

Last time we had a function
$$f(x) = 1000 + \frac{x}{2}$$
, which we called informally a contraction.

More formally, let $C = [0, 10^6]$.

Note $K(x) \in C$ for all $x \in C$, so it is a mapping $K: C \subseteq X \to C$.

The contraction constant $\theta = \frac{1}{2}$ because
$$|f(x) - f(y)| = \left|1000 + \frac{x}{2} - \left(1000 + \frac{y}{2}\right)\right| = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y|.$$

$$|f(x)-f(y)| = \left|1000+\frac{x}{2} - \left(1000+\frac{y}{2}\right)\right| = \left|\frac{x}{2}-\frac{y}{2}\right| = \frac{1}{2}|x-y|.$$

So $f(x) = 1000 + \frac{x}{2}$ is a contraction on $C = [0, 10^6]$.

What about $C = [0, 10]$?

NO. Because $f(0) = 1000 \notin [0, 10]$.

Last time, we had a function

$$f(x) = \sqrt{x}.$$

Is it a contraction on $C = [0, 1]$?

No, because $|\sqrt{0} - \sqrt{1}| = |0 - 1|$, so it doesn't contract them.

What about on $C = [0.5, 1.5]$?

Yes. First note that $0.5 \le \sqrt{x} \le 1.5$ if $x \in [0.5, 1.5]$.

Also, For $x \ge 0.5$, $\frac{1}{\sqrt{x}+\sqrt{y}} \le \frac{1}{\sqrt{0.5}+\sqrt{0.5}} = \frac{1}{2\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2}} \approx 0.707$.

$$\text{Now } |\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \le \frac{1}{\sqrt{2}}|x-y|.$$

Thus, $f(x) = \sqrt{x}$ is a contraction on $C = [0.5, 1.5]$.

When checking a contraction, make sure to check that the mapping
(1) puts points in a subset C closer together by a multiplicative factor.
(2) maps into the subset C in question.

Theorem: Banach fixed-point theorem (contractive principle)
(Teschl 2.1)

Let C be a (nonempty) closed subset of a Banach space X and let $K : C \to C$ be a contraction. Then K has a unique fixed point $\bar{x} \in C$ s.t.

$$\|K^n(x) - \bar{x}\| \le \frac{\theta^n}{1-\theta}\|K(x) - x\|, \quad x \in C.$$

proof. Existence Fix $x_0 \in C$ and consider the sequence

$$x_n = K^n(x_0).$$

Then we have $\|x_{n+1} - x_n\| \le \theta \|x_n - x_{n-1}\|$

$\le \theta^2 \| \quad \|$

Then we have $\|x_{n+1} - x_n\| \le \Theta \|x_n - x_{n-1}\|$

$$\le \Theta^2 \|x_{n-1} - x_{n-2}\|$$
$$\vdots$$
$$\le \Theta^n \|x_1 - x_0\|.$$

By the triangle inequality (since we have a metric), for $n > m$

$$\|x_n - x_m\| = \|(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots (x_{m+1} - x_m)\|$$
$$\le \|x_n - x_{n-1}\| + \cdots + \|x_{m+1} - x_m\|$$
$$\le \Theta^{n-1}\|x_1 - x_0\| + \cdots + \Theta^m \|x_1 - x_0\|$$
$$= \Theta^m \sum_{j=0}^{n-m-1} \Theta^j \|x_1 - x_0\|$$
$$= \Theta^m \cdot \frac{1 - \Theta^{n-m}}{1 - \Theta} \|x_1 - x_0\| \le \frac{\Theta^m}{1 - \Theta} \|x_1 - x_0\|.$$

Because for any ε, we can find an M s.t. $\forall\, n, m \ge M$, $\|x_n - x_m\| \le \varepsilon$, this sequence is Cauchy.

Because we are in a Banach space, Cauchy sequences tend to some limit, which here we call $\bar{x}$.

Note, $\|K(\bar{x}) - \bar{x}\| = \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$, so $\bar{x}$ is a fixed point.

<u>Uniqueness</u>: Let $\bar{x}$ and $\bar{y}$ be fixed points of K, $\bar{x}, \bar{y} \in C$.

Then $\|K(\bar{x}) - K(\bar{y})\| = \|\bar{x} - \bar{y}\|$.

But $\|K(\bar{x}) - K(\bar{y})\| \le \Theta \|\bar{x} - \bar{y}\|$ because K is a contraction.

So $\|\bar{x} - \bar{y}\| \le \Theta \|\bar{x} - \bar{y}\|$.

$\Theta < 1$, so this can only be true if $\|\bar{x} - \bar{y}\| = 0$

$\Rightarrow \bar{x} = \bar{y}$. $\square$

We have now proven that so long as we have a contraction in a closed subset of a Banach space, there exists a unique fixed point.

Let's apply the contraction principle to some examples

Ex. Consider a mapping $K: \mathbb{R} \to \mathbb{R}$, defined by $K(x) = 1000 + \frac{x}{2}$. Prove that for any $x \in \mathbb{R}$, $\lim_{n\to\infty} K^n(x) = 2000$.

In order to use the contraction principle, we need a closed subset $C \subseteq \mathbb{R}$ s.t. K is a contraction on $\mathbb{R}$.

$\mathbb{R}$ is a closed subset of $\mathbb{R}$, and $|K(x) - K(y)| \le \frac{1}{2}|x-y|$ for all $x, y \in \mathbb{R}$.

So there exists only one fixed point.

Note $K(2000) = 2000$, so $\bar{x} = 2000$ is a fixed point.

Thus, all starting points converge to 2000 under repeated iterations of K.

Ex. Consider a mapping $K: \mathbb{R} \to \mathbb{R}$, $K(x) = \sqrt{x}$.

Prove that for any $x_0 \ge 0.5$ $\lim_{n\to\infty} K^n(x_0) = 1$.

In order to use the contraction principle, we need a closed subset of $\mathbb{R}$ on which K is a contraction.

Obviously, the entire real line doesn't work.

Lemma: If $f: C \subseteq \mathbb{R} \to \mathbb{C}$ has a continuous derivative on a closed interval C, and $\forall x$, $|f'(x)| \le \theta$ for some constant $\theta < 1$, then f is a contraction.

proof.: Let $x, y \in \mathbb{R}$. By the Mean Value Theorem, $\exists c \in [x, y]$ s.t. $|f(x) - f(y)| = |f'(c)|\,|x-y|$.

But $f'(c) \le \theta < 1$, so $|f(x) - f(y)| \le \theta\,|x-y|$, completing the proof. $\square$

Back to $K(x) = \sqrt{x}$. Note that $K'(x) = \frac{1}{2\sqrt{x}}$.

Thus $|K'(x)| = \frac{1}{2\sqrt{x}} \le \frac{1}{2\sqrt{\frac{1}{2}}}$ for all $x \ge \frac{1}{2}$

$\Rightarrow |K'(x)| \le \frac{\sqrt{2}}{2} \approx 0.707$ for all $x \ge \frac{1}{2}$.

Also $K(x) \in [0.5, \infty)$ for all $x \in [0.5, \infty)$. Thus, $K(x)$ is a contraction.

Since $K(1) = 1$ is a fixed pt., $\lim_{n\to\infty} K^n(x) = 1$ for all $x \ge 0.5$.

Recall: We proved earlier that $C(I)$ is a Banach space of functions

Now we just need to find a contraction whose fixed point is the solution to an ODE.

Picard Iteration:

Consider an initial value problem (IVP)

$$\dot{x} = f(t, x), \quad x(t_0) = x_0,$$

where $x, t \in \mathbb{R}$, and $f \in C(U, \mathbb{R})$, where $U \subseteq \mathbb{R}^2$ is an open subset of $\mathbb{R}^2$ and $(t_0, x_0) \in U$.

Note: We consider here $x \in \mathbb{R}$. The proof we give also works with only minor modifications for $x \in \mathbb{R}^n$, $f \in C(U, \mathbb{R}^n)$, where $U \subseteq \mathbb{R}^{n+1}$. This means that existence and uniqueness will hold for all first-order systems. Because we showed at the beginning of class that all higher-order systems can be transformed to equivalent first-order system, this proof will apply to all ODEs under certain technical conditions.

Let's integrate both sides with respect to t:

$$\int_{s=t_0}^{s=t} \dot{x}(s)ds = \int_{s=t_0}^{s=t} f(s, x(s))ds$$

$$x(t) - x(t_0) = \int_{t_0}^{t} f(s, x(s))ds$$

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x(s))ds$$

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

This integral equation is equivalent to our original ODE.

i.e. if for some function $x(t)$, $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$, then $x(t)$ is a solution to the original $\dot{x} = f(t, x)$.

Let's define Picard iteration by a map $K: C(U, \mathbb{R}) \to C(U, \mathbb{R})$

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

And the Picard Iterates

$$x_0(t) = x_0 \qquad \text{(the constant function through the scalar } x_0\text{)}$$

$$x_1(t) = K(x_0)(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))ds$$

$$x_1(t) = K(x_0)(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\,ds$$

$$x_2(t) = K^2(x_0)(t) = K(x_1)(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\,ds$$

$$x_3(t) = K^3(x_0)(t) = K(x_2)(t) = x_0 + \int_{t_0}^{t} f(s, x_2(s))\,ds$$

$$\vdots$$

$$x_m(t) = K^m(x_0)(t) = K(x_{m-1})(t) = x_0 + \int_{t_0}^{t} f(s, x_{m-1}(s))\,ds.$$

The solution $x(t)$ is a fixed point under Picard iteration, so if we can prove Picard iteration to be a contraction, then that would prove existence and uniqueness for the solution to the ODE.

But first, let's try some examples of Picard iteration.

Ex. $\dot{x}(t) = tx$, and $x(0) = 1$.

Note, the real solution (by sep. of variables) is $x(t) = e^{t^2/2}$.

The Picard iterates are (letting $f(t,x) = tx$)

$$x_0(t) = x_0 = 1.$$

$$x_1(t) = 1 + \int_0^t s\, x_0(s)\,ds = 1 + \int_0^t s\,ds = 1 + \frac{t^2}{2}$$

$$x_2(t) = 1 + \int_0^t s\left(1 + \frac{s^2}{2}\right)ds = 1 + \frac{t^2}{2} + \frac{t^4}{8}$$

$$x_3(t) = 1 + \int_0^t s\left(1 + \frac{s^2}{2} + \frac{s^4}{8}\right)ds = 1 + \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48}.$$

Note that $e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{3!} + \frac{a^4}{4!} + \dots$

So $e^{t^2/2} = 1 + \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48} + \frac{t^8}{384} + \dots$

So the Picard iterates are slowly approximating the true solution.

Next time: We'll prove that Picard iteration is a contraction under certain circumstances.